Tetragonal or hexagonal symmetry in modeling of failure criteria for transversely isotropic materials

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Abstract

Present work deals with modeling of failure criteria for transversely isotropic materials. Analysis comprises two classes of symmetry: classical Tsai-Wu tetragonal and new Tsai-Wu based hexagonal. Detail analysis of both classes of symmetry with respect to their advantages as well as limitations is presented. Finally, simple comparison of differences between limit curves corresponding to cross sections by plane of transverse isotropy is done.

Keywords

Transverse isotropy, tetragonal and hexagonal symmetry, Tsai-Wu criterion

1. Introduction

In a general case of brittle materials that exhibit anisotropy (e.g. concrete, ceramic materials, rocks, composite materials, etc.) and tension-compression asymmetry a transversely isotropic Tsai-Wu criterion of initial failure [6] is applicable

$$F[(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2]+H(\sigma_x-\sigma_y)^2+2L\tau_{xy}^2+2M(\tau_{yz}^2+\tau_{zx}^2)+P(\sigma_x-\sigma_y)+R\sigma_z=1 \tag{1}$$

Eq. (1) contains 5 independent material coefficients referring to appropriate tensile and compressive strengths k_{tx}, k_{cx}, k_{tz}, k_{cz} and one shear strength k_{zx}, hence, in order to calibrate it following tests must be performed

$$\begin{array}{lll}\sigma_x=k_{tx} & \to & (F+H)k_{tx}^2+Pk_{tx}=1\\ \sigma_x=-k_{cx} & \to & (F+H)k_{cx}^2-Pk_{cx}=1\\ \sigma_z=k_{tz} & \to & 2Fk_{tz}^2+Rk_{tz}=1\\ \sigma_z=-k_{cz} & \to & 2Fk_{cz}^2-Rk_{cz}=1\\ \tau_{zx}=k_{zx} & \to & 2Mk_{zx}^2=1\end{array} \tag{2}$$

Solution of Eq. (2) with respect to F,H,M,P and R takes the form

$$\begin{array}{lll}F=\dfrac{1}{2k_{tz}k_{cz}} & H=\dfrac{1}{k_{tx}k_{cx}}-\dfrac{1}{2k_{tz}k_{cz}} & M=\dfrac{1}{k_{zx}^2}\\ P=\dfrac{1}{k_{tx}}-\dfrac{1}{k_{cx}} & R=\dfrac{1}{k_{tz}}-\dfrac{1}{k_{cz}} & \end{array} \tag{3}$$

Magnitude of material coefficient L, referring to shear strength in plane of transverse isotropy, is not independent and can be calculated from known relation (see Chen and Han [1], Ganczarski and Skrzypek [3])

$$2L=2(F+2H)=\frac{4}{k_{tx}k_{cx}}-\frac{1}{k_{tz}k_{cz}} \tag{4}$$

Hence, after substitution of Eqs. (3) and (4) to Eq. (1) one can get final form of the transversely isotropic Tsai-Wu criterion

$$\frac{\sigma_x^2+\sigma_y^2}{k_{tx}k_{cx}}+\frac{\sigma_z^2}{k_{tz}k_{cz}}-\left(\frac{2}{k_{tx}k_{cx}}-\frac{1}{k_{tz}k_{cz}}\right)\sigma_x\sigma_y-\frac{(\sigma_y+\sigma_x)\sigma_z}{k_{tz}k_{cz}}+\left(\frac{4}{k_{tx}k_{cx}}-\frac{1}{k_{tz}k_{cz}}\right)\tau_{xy}^2+$$
$$\frac{\tau_{yz}^2+\tau_{zx}^2}{k_{zx}^2}+\left(\frac{1}{k_{tx}}-\frac{1}{k_{cx}}\right)(\sigma_x+\sigma_y)+\left(\frac{1}{k_{tz}}-\frac{1}{k_{cz}}\right)\sigma_z=1 \tag{5}$$

It is obvious that material coefficients in plane of transverse isotropy that precede terms $\sigma_x\sigma_y$ and τ_{xy} are not fully independent since they contain not only in plane tensile and compressive stresses k_{tx}, k_{cx} but also out of plane tensile and compressive stresses k_{tz}, k_{cz}. Consequently, Eq. (5) can be classified as the tetragonal transversely isotropic Tsai-Wu criterion of initial failure.

2. Convexity loss in case of high orthotropy

Applicability range of Tsai-Wu orthotropic criterion (5) to properly describe initiation of failure in some engineering materials that exhibit high orthotropy degree, is bounded by a possible ellipticity loss of the limit surface. Other words, a physically inadmissible degeneration of the single convex and simply connected ellipticital limit surface into two concave hyperbolas surfaces occur. The following inequality bounds the range of applicability for Tsai-Wu criterion (see Ottosen and Ristinmaa [4], Ganczarski and Skrzypek [2])

$$\frac{1}{k_{tz}k_{cz}}\left(\frac{4}{k_{tx}k_{cx}}-\frac{1}{k_{tz}k_{cz}}\right)>0 \tag{6}$$

Substitution of the dimensionless parameter $R=2(k_{tz}k_{cz}/k_{tx}k_{cx})-1$(extension of Hosford and Backofen parameter [3]), leads to the simplified restriction

$$R>-0.5 \tag{7}$$

If the above inequalities (6)–(7) do not hold, elliptic cross sections of the limit surface degenerate to two hyperbolic branches and the lack of convexity occurs. To illustrate this limitation, the yield curves in two planes: the transverse isotropy (σ_x,σ_y)

$$\sigma_x^2-\frac{2R}{1+R}\sigma_x\sigma_y+\sigma_y^2+(k_{cx}-k_{tx})(\sigma_x+\sigma_y)=k_{tx}k_{cx} \tag{8}$$

and the orthotropy plane (σ_x,σ_z)

$$\sigma_x^2-\frac{2}{1+R}\sigma_x\sigma_z+\frac{2}{1+R}\sigma_z^2+(k_{cx}-k_{tx})\sigma_x+k_{tz}k_{cz}\sigma_z=k_{tx}k_{cx} \tag{9}$$

for various R-values, are sketched in Fig. 1a, b respectively. It is observed that when R, starting from $R=3$, approaches the limit $R=-0.5$, the curves change from closed ellipses to two parallel lines or parabola, whereas for $R<-0.5$, concave hyperbolas appear.

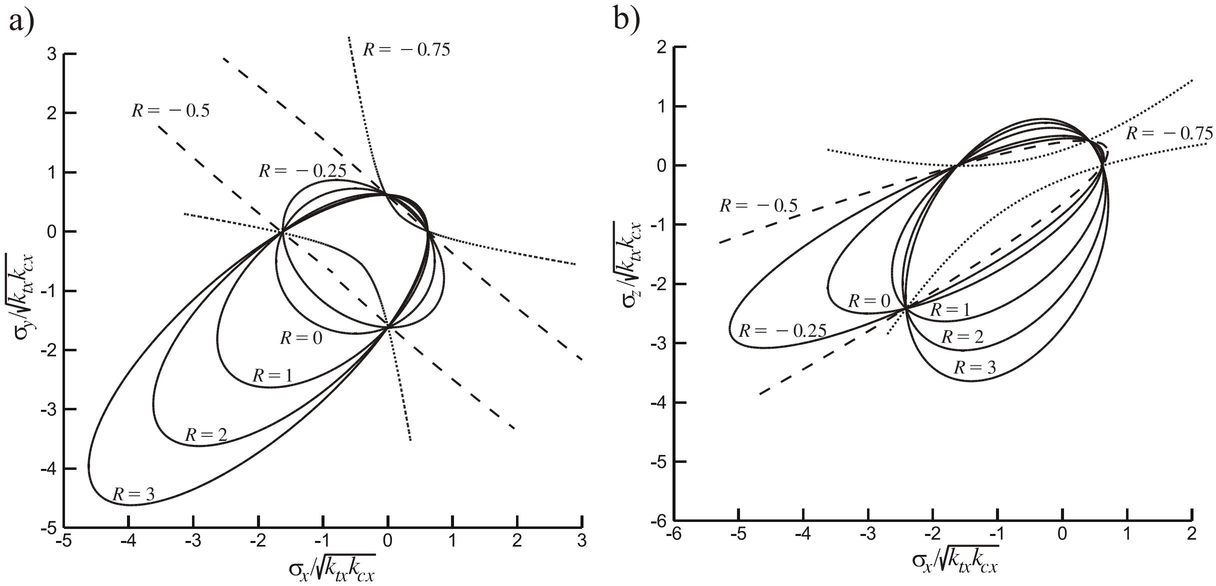

Fig. 1. *Convexity loss of tetragonal Tsai-Wu criterion in case of high orthotropy degree for traverse isotropy: a) plane of transverse isotropy (σ_x, σ_y), b) plane of orthotropy (σ_x, σ_z).*

3. Modified Tsai-Wu based hexagonal failure criterion

Except the tetragonal transversely isotropic Tsai-Wu criterion Eq. (5) one can introduce hexagonally isotropic Tsai-Wu failure criterion

$$\begin{aligned} &\frac{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2}{k_{tx}k_{cx}} + \frac{\sigma_z^2}{k_{tz}k_{cz}} - \frac{(\sigma_y + \sigma_x)\sigma_z}{k_{tz}k_{cz}} + \frac{3\tau_{xy}^2}{k_{tx}k_{cx}} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \\ &\left(\frac{1}{k_{tx}} - \frac{1}{k_{cx}}\right)(\sigma_x + \sigma_y) + \left(\frac{1}{k_{tz}} - \frac{1}{k_{cz}}\right)\sigma_z = 1 \end{aligned} \tag{10}$$

in which coefficients that precede terms $\sigma_x\sigma_y$ and τ_{xy} are allways positive. These prevent elliptic form of failure curves from „cracking“ and reduce (10) to the Huber-von Mises-Hencky ellipse “shifted” outside the origin of co-ordinate system in case of transverse isotropy plane.

4. Results

Both the Tsai-Wu transversely isotropic initial failure criteria: tetragonal Eq. (5) and new hexagonal type (10) are compared for columnar ice, the experimental data of which was established by Ralston [5] in Tab. 1 in plane of transverse isotropy (σ_x, σ_y), shear plane (σ_x, τ_{xy}) and in plane of orthotropy (σ_x, σ_z) Fig. 2. Subsequent cross sections of the limit surface are ellipses, that exhibit strong oblatness in case tetragonal symmetry, the centers of which are shifted outside the origin of co-ordinate system towards the quarter referring to compressive stresses. In case of cross section by plane of transverse isotropy (see Fig. 2a) the symmetry axis has obviously inclination equal 45^{o} to the axes of co-ordinate system, otherwords it overlaps projection of hydrostatic axis at the transverse isotropy plane (σ_x, σ_y), contrary to the cross section by plane of orthotropy (see Fig. 2b) the main semi-axis of ellipse is inclined by 71.1^{o}. It has to be emphasize that in case of columnar ice compressive strength along othotropy axis k_{cz} is over 10 times greater than tensile strength k_{tz}, whereas analogous ratio k_{cx}/k_{tx} is approximately equal to 7 in case of transverse isotropy plane.

Table 1. – *Experimental data for columnar ice (after Ralston [5]).*

Tensile strength		Compressive strength	
k_{tx}	1.01 MPa	k_{cx}	7.11 MPa
k_{tz}	1.21 MPa	k_{cz}	13.5 MPa

Moreover, ratio of semi-axes for Tsai-Wu tetragonal ellipse in (σ_x, σ_y) plane essentially exceeds analogous ratio for Huber-von Mises-Hencky ellipse, contrary to the case of Tsai-Wu hexagonal ellipse.

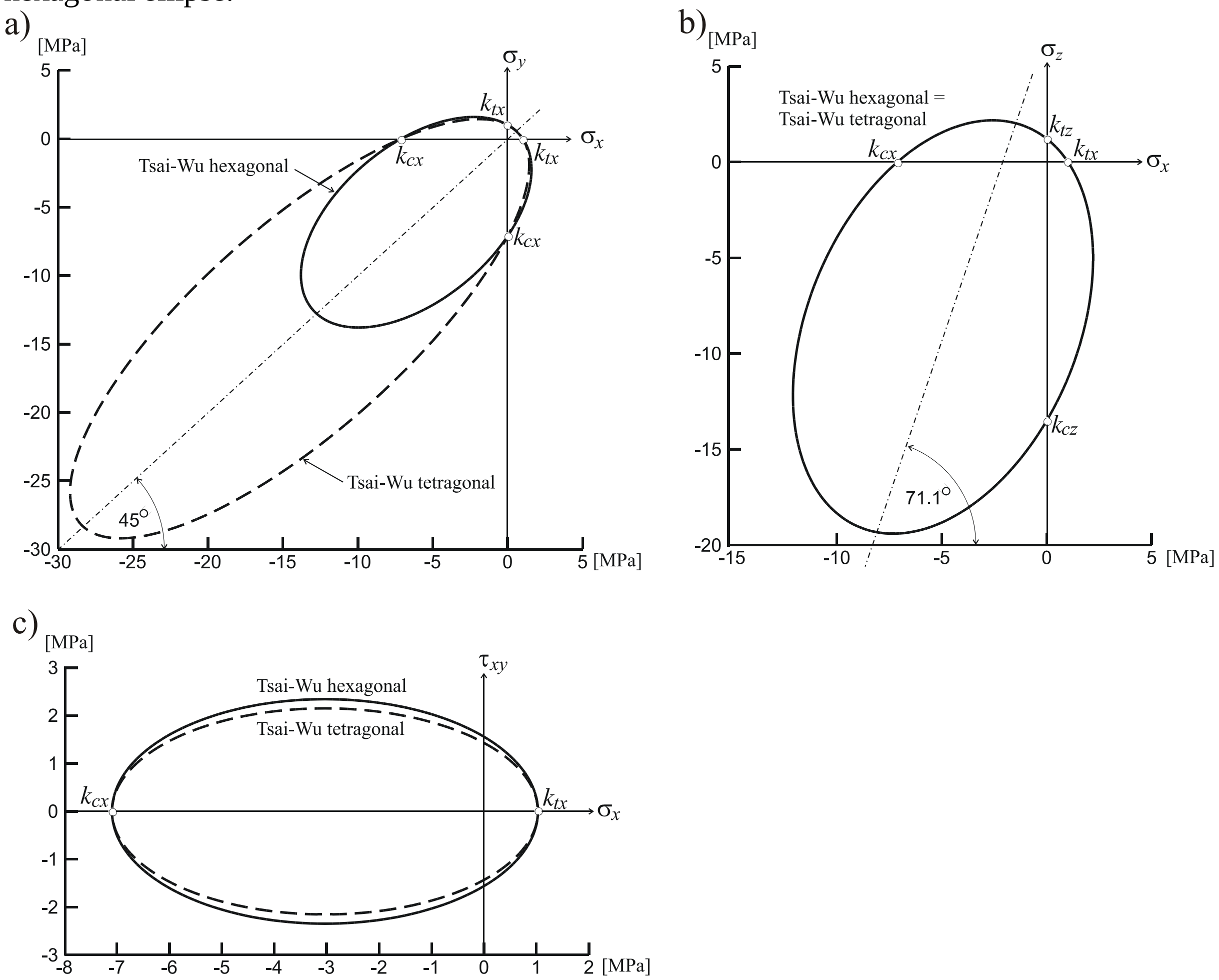

Fig. 2. *Comparison of transversely isotropic Tsai-Wu initial failure of tetragonal and hexagonal types for columnar ice in case of: a) plane of traverse isotropy* (σ_x, σ_y)*, b) plane of orthotropy* (σ_x, σ_z)*, c) shear plane* (σ_x, τ_{xy})*.*

It is also worth to emphasize that although the tetragonal transversely isotropic Tsai-Wu failure criterion Eq. (5) and the hexagonal transversely isotropic Tsai-Wu failure criterion Eq. (10) contain the same number of 5 independent strengths k_{tx}, k_{cx}, k_{tz}, k_{cz} and k_{zx}, only criterion (10) is free from convexity loss and simultaneously truly transversely isotropic in sense of hexagonal class of symmetry.

4. Conclusion

Both transversely isotropic failure criteria: Tsai-Wu tetragonal Eq. (5) and Tsai-Wu hexagonal Eq. (10) perform cylindrical surfaces in the space of principal stresses. However, the tetragonal criterion is represented by elliptic cylinder the axis of which coincides with the hydrostatic axis contrary to the hexagonal one, which represents elliptic cylinder the axis of which is not equally inclined to principal stresses. Hence, hexagonal Tsai-Wu failure criterion does not satisfies deviatoric property, that is a consequence of its coincidence with Huber-von Mises-Hencky criterion in the plane of transverse isotropy as well as a property of saving cylindrical nature despite of high ratio of orthotropy. Choice of appropriate transversely yield criterion either tetragonal Eq. (5) or hexagonal Eq. (10) depends on coincidence with experimental tests done on real materials, that can subjected one or other class of symmetry, but also can exhibit properties different than aforementioned cases. Key point for proper classification of real transversely isotropic material to one of symmetry class (tetragonal, hexagonal) is the shape of limit curve belonging to the plane of transverse isotropy.

Symbols

σ_x	stress along *x* axis	(MPa)
σ_y	stress along *y* axis	(MPa)
σ_z	stress along z axis	(MPa)
τ_{xy}	shear stress in *xy* plane	(MPa)
τ_{yz}	shear stress in *yz* plane	(MPa)
τ_{zx}	shear stress in *zx* plane	(MPa)
k_{tx}	tensile strength along *x* axis	(MPa)
k_{cx}	compressive strength along *x* axis	(MPa)
k_{tz}	tensile strength along *z* axis	(MPa)
k_{cz}	compressive strength along *z* axis	(MPa)
k_{zx}	shear strength in *xz* plane	(MPa)

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